Semester 2 Examination 2016
Question/Answer booklet

Year 12
MATHEMATICS SPECIALIST
Book 1 of 2
Section One
(Calculator-free)

Circle teacher's name Mr Hill Mr Lau

Time allowed for this section
Reading time before commencing work: 5 minutes
Working time for section: 50 minutes

Material required/recommended for this section
To be provided by the supervisor
This Question/Answer booklet for Section One, and a separate formula sheet which may also be used for Section Two.

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, highlighter, eraser, ruler

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this examination

	Number of questions available	Number of questions to be attempted	Working time (minutes)	Marks available	Percentage of Exam
Section One Calculator—free	**8**	**8**	**50**	**53**	**35%**
Section Two Calculator—assumed	13	13	100	99	65%
			Total marks	152	100%

Instructions to candidates

1. Answer all the questions in the spaces provided.

2. Spare answer pages are provided at the end of this booklet. If you need to use them, indicate in the original answer space where the answer is continued i.e. give the page number.

3. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Correct answers given without supporting reasoning may not be allocated full marks. Incorrect answers given without supporting reasoning cannot be allocated any marks. If you repeat an answer to any question, ensure that you cancel the answers you do not wish to have marked.

4. It is recommended that you **do not use pencil** except in diagrams.

Question 1 **(5 marks)**

(a) Evaluate $(2 - i)(1 - 2i)$, leaving your answer in polar form. (2 marks)

(b) If $z_1 = \text{cis}\,\dfrac{\pi}{6}$, $z_2 = r\,\text{cis}\,\theta$ and $z_1 z_2$ is a real number, determine the value(s) of θ where $-\pi < \theta \leq \pi$. (3 marks)

Question 2 **(7 marks)**

(a) Two random samples of 9 students from Year 12 Hale School are taken. The following table shows their brief summary of statistics of their heights.

	Mean (cm)	Standard deviation (cm)
Sample 1	177.5	10.5
Sample 2	179.2	9.4

Briefly explain why the two sample means are expected to be different. (2 marks)

(b) The sampling distribution of the heights of random samples of 9 students from Year 12 Hale School has a mean of 178.4 cm and a standard deviation of 8.2 cm.

(i) Find the mean and standard deviation of **all** Year 12 Hale students' height. (3 marks)

(ii) Is it valid to assume that the sampling distribution of the heights is approximately normal? Explain. (2 marks)

Question 3 **(5 marks)**

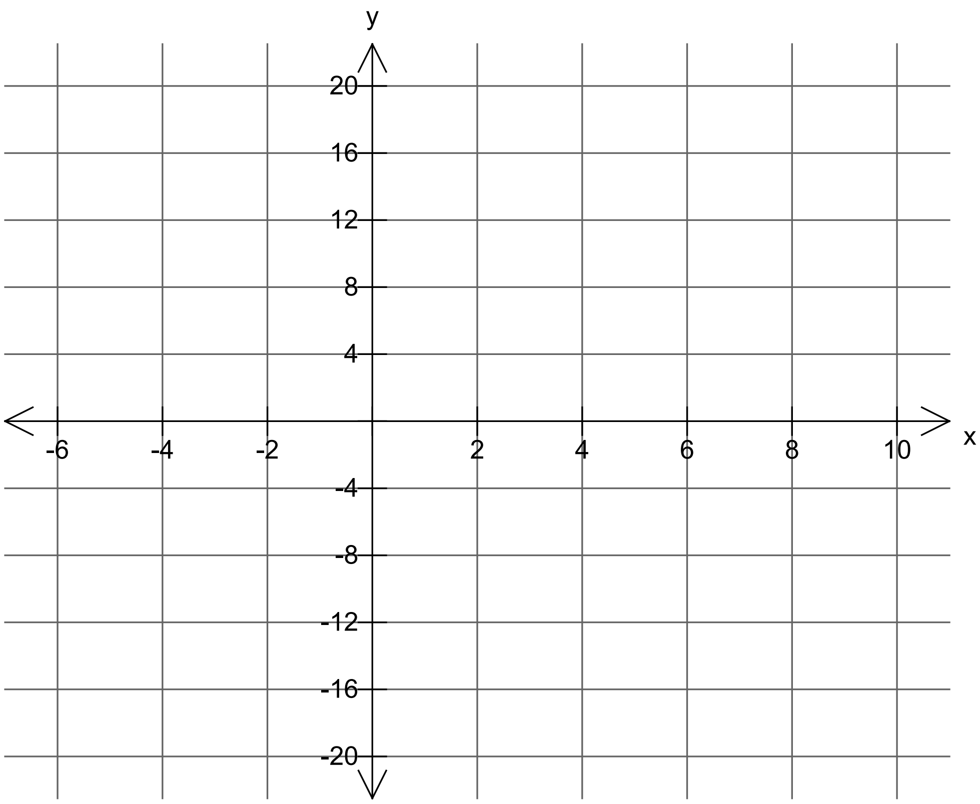
(a) Determine the value of c if the three equations $\begin{cases} x + 2y - z = 0 \\ 2x - y + z = -1 \\ -x + y + cz = 1 \end{cases}$ have **no** solutions. (4 marks)

(b) Given the answer in (a), give a geometric interpretation of the three planes in (a). (1 mark)

Question 4 **(6 marks)**

(a) Simplify the function $y = \dfrac{-x^2 + 7x - 18}{x - 3}$ to the form $y = (ax + b) - \dfrac{6}{x - 3}$. (2 marks)

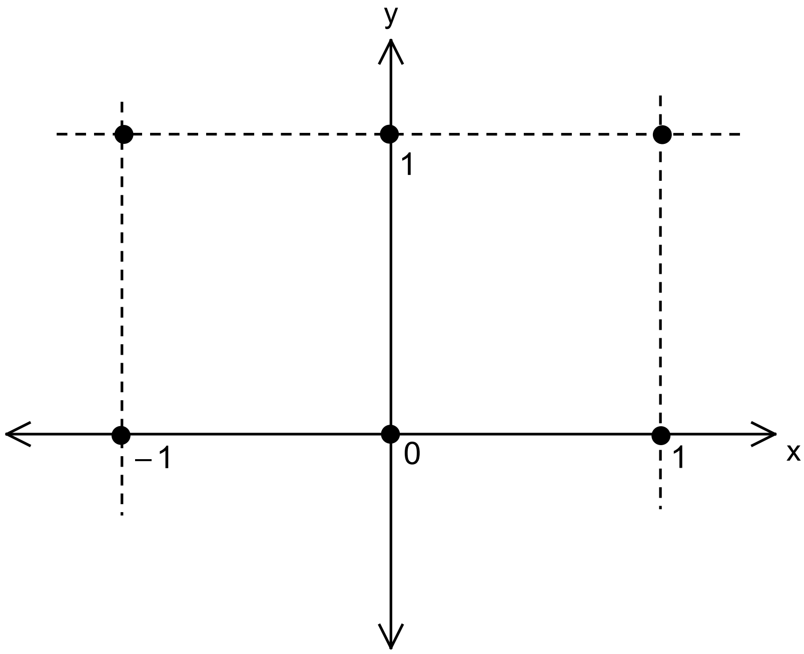
(b) Hence sketch the graph of the function $y = \dfrac{-x^2 + 7x - 18}{x - 3}$. (4 marks)



Question 5 **(8 marks)**

(a) Solve $\dfrac{dy}{dx} = 4x\sqrt{y}$ if y = 1 when x = -1. (4 marks)

(b) Sketch the slope fields of $\dfrac{dy}{dx} = 4x\sqrt{y}$ for the 6 points indicated below. (4 marks)



Question 6 **(11 marks)**

Find the following integrals.

(a) $\int \cos^2 2x \, dx$ (3 marks)

(b) $\int \frac{3x+7}{x^2+3x-4} \, dx$ (3 marks)

(c) $\int \frac{2+\ln x}{x} dx$ where $x > 0$ (2 marks)

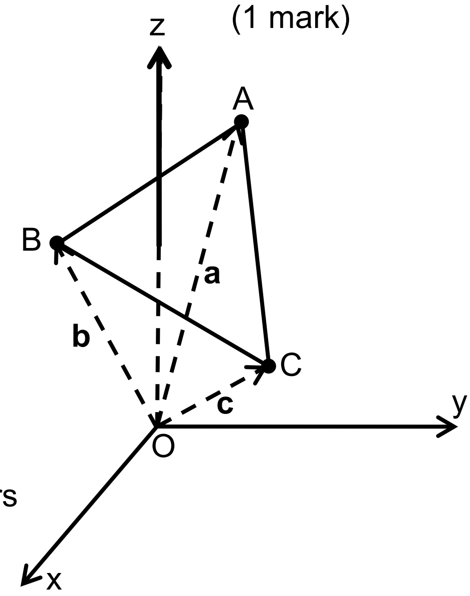
(d) $\int \frac{-1}{1+e^x} dx$ (Hint: $a^b = \frac{1}{a^{-b}}$) (3 marks)

Question 7 **(5 marks)**

(a) Give a geometric interpretation of the magnitude of $\mathbf{a} \times \mathbf{b}$ regarding the parallelogram spanned by the vectors $\mathbf{a}$ and $\mathbf{b}$. (1 mark)

(b) Consider 3 points A, B and C with corresponding position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in space as shown below. The points O, A, B and C form a tetrahedron.

(i) Express **AB** and **AC** in terms of $\mathbf{a}$, $\mathbf{b}$ and/or $\mathbf{c}$. (1 mark)



(ii) The area vector of a face is a vector perpendicular to the face, pointing **outwards**, whose magnitude is the area of the face.
Show that the sum of the outward pointing area vectors of the 4 faces of the tetrahedron equals the zero vector.

(3 marks)

Question 8 **(6 marks)**

A model for a population, P, of numbats is $P = \dfrac{900}{3 + 2e^{-\frac{t}{4}}}$, where t is the time in years from the start of 2015.

(a) What is the initial population? (1 mark)

(b) What is the predicted long term population? (1 mark)

(c) Show **clearly** that P satisfies the differential equation $\dfrac{dP}{dt} = \dfrac{P}{4}(1 - \dfrac{P}{300})$. (4 marks)

Additional working space

Question number(s): ________________

Semester 2 Examination 2016

Question/Answer booklet

HALE SCHOOL

Year 12
MATHEMATICS SPECIALIST
Book 2 of 2
Section Two
(Calculator-assumed)

Circle teacher's name Mr Hill Mr Lau

Time allowed for this section

Reading time before commencing work: 10 minutes
Working time for section: 100 minutes

Material required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet for Section Two. Candidates may use the separate formula sheet from Section One.

To be provided by the candidate

Standard items: pens, pencils, pencil sharpener, highlighter, eraser, ruler

Special items: drawing instruments, templates, notes on up to two unfolded sheets of A4 paper, and up to three calculators, approved for use in this examination.

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Question 9 **(6 marks)**

Suppose that the heights of students at Hale School follow a normal distribution. It is found that a large random sample of Hale students has a mean of 172 cm with a 6 cm margin of error at a confidence level of 90%.

(a) Write down the confidence interval for estimating the mean height of Hale School students. Interpret your answer. (2 marks)

(b) What would be the change in the confidence interval (increase, decrease or no change) if the sample size increases? Explain. (2 marks)

(c) What would be the change in the confidence interval if the confidence level is set at 95% instead of 90%? Explain. (2 marks)

Question 10 **(7 marks)**

Consider the functions $f(x) = \ln(5e - x)$ and $g(x) = e^x + 4e$.

(a) Give an example of x such that $f(g(x))$ is **not** defined. (1 mark)

(b) (i) State the largest restricted domain of $g(x)$ such that $f(g(x))$ is a function. (2 marks)

Consider the restricted domain of $g(x)$ in part (b) (i).

(ii) Show that $f(g(x))$ is a one-to-one function. (4 marks)

Question 11 **(8 marks)**

Random samples of size 36 are drawn from a large population of ball bearings.
A particular sample has a mean of 22.40 g, with a standard deviation of 0.72 g.

(a) Estimate the standard deviation of the sampling distribution of the mean. (2 marks)

Consider a 96% confidence interval for the population mean.

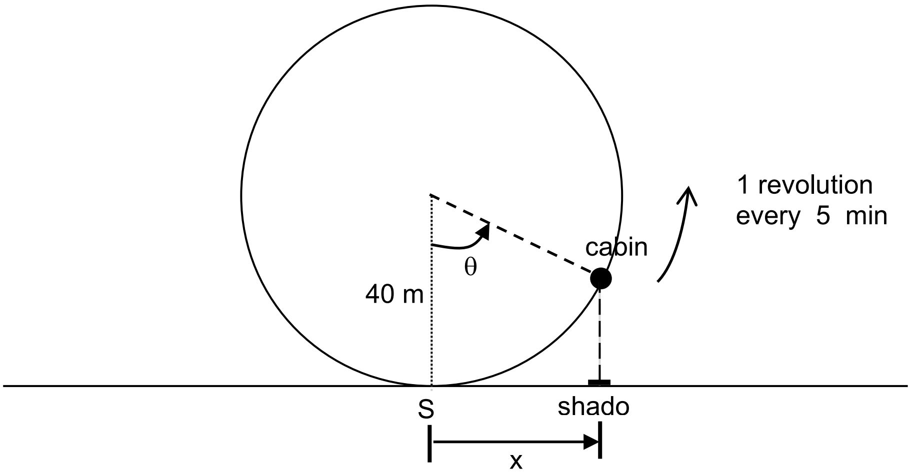
(b) State the values (correct to 4 decimal places) of the standard normal variable corresponding to 96% confidence interval. (1 mark)

(c) Determine the 96% confidence interval for the population mean. (2 marks)

(d) Assume that the population standard deviation of ball bearings can be estimated by 0.72 g, determine how large a sample must be taken in order to be 96% confident that the error in the estimation of population mean will not exceed 0.15 g. (3 marks)

Question 12 **(8 marks)**

Laura is travelling on a Ferris wheel of radius 40 metres, that is turning at a constant angular speed of one revolution every 5 minutes. Initially, Laura's cabin is at ground level, at point S. The sun is directly overhead and casts a shadow on the ground directly below Laura's cabin.



Let θ be the angle that the radius joining the centre of the Ferris wheel to the cabin makes with the downward vertical, and

x be the **horizontal** displacement of the shadow from the point S on the ground.

(a) Express θ in terms of time t (in min). (2 marks)

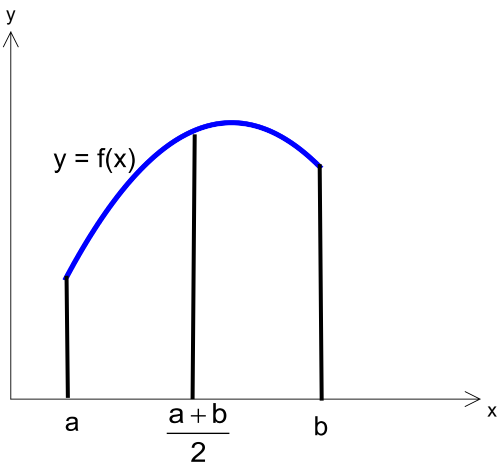
(b) Show that the **shadow** moves with simple harmonic motion. (3 marks)

(c) Find the exact speed at which the shadow is moving horizontally, when the cabin is 20 m above the ground. (3 marks)

Question 13 **(9 marks)**

The area A under the graph of $y = f(x)$ from $x = a$ to $x = b$ can be approximated by

$$A \approx \frac{b-a}{6}[f(a)+4f(\frac{a+b}{2})+f(b)].$$



The above method is used to estimate the area under the graph of $y = x \ln x$ from $x = 2$ to $x = 3$.

Use 4 strips to approximate the area and give your answers correct to 4 decimal places for parts (a) and (b), if necessary.

(a) Complete the following table. (2 marks)

x	2				3
f(x)					

(b) Show **clearly** how you use the above formula to obtain an estimate of the required area. (4 marks)

(c) State an approximation of the area correct to **6** decimal places. Provide evidence for your estimation. (2 marks)

(d) State the **type** of function $f(x)$ such that the formula

$$A \approx \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)]$$

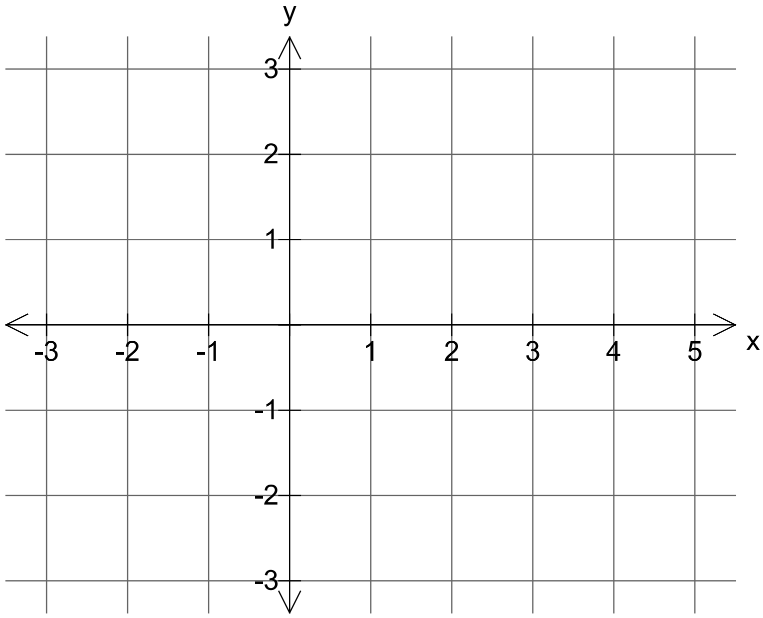
actually gives an exact answer. (1 mark)

Question 14 **(6 marks)**

Let $z = x + yi$ be a complex number where x and y are real numbers.

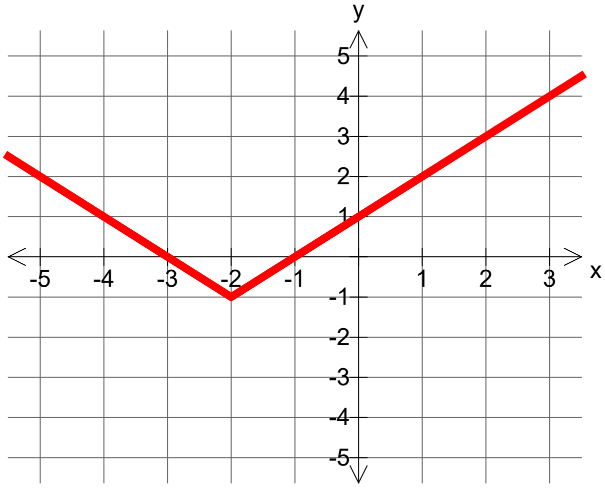
(a) Show that $\dfrac{z-2i}{z+2i} = \dfrac{(x^2+y^2-4)-4xi}{x^2+(y+2)^2}$. (2 marks)

(b) Sketch the locus of z such that $\arg\left(\dfrac{z-2i}{z+2i}\right) = \dfrac{3\pi}{4}$. (4 marks)

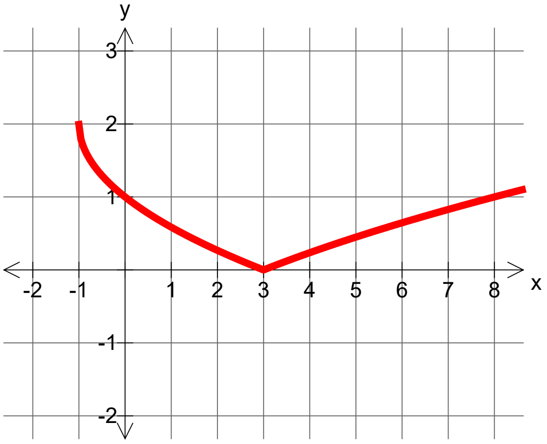


Question 15 **(7 marks)**

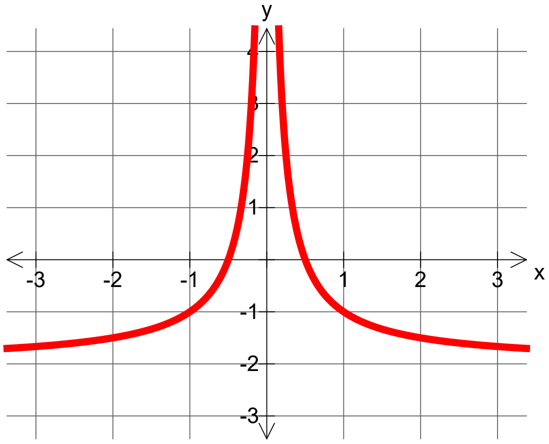
(a) Given the graph of the function $y = f(x)$ as shown below, sketch the graph of its reciprocal function $y = \dfrac{1}{f(x)}$. (3 marks)



(b) Determine the function $y = g(x)$, where $g(x)$ is a square-root function, such that the graph of $y = |\,g(x)\,|$ is as shown below. (2 marks)



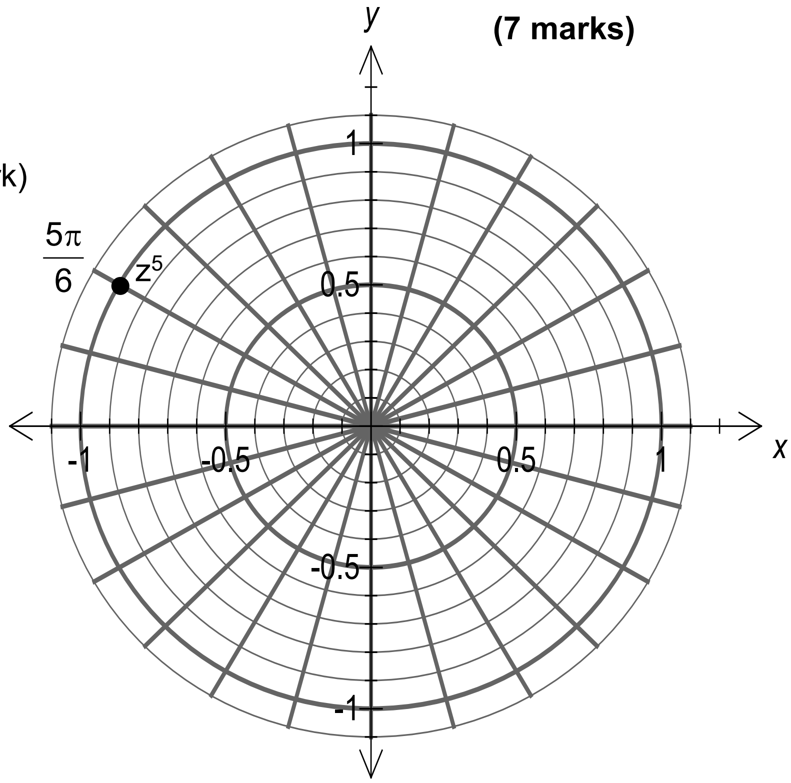
(c) Determine the function $y = h(x)$, where $h(x)$ is a reciprocal function, such that the graph of $y = h(\,|x|\,)$ is as shown below. (2 marks)



Question 16 **(7 marks)**

A complex number z^5 is shown on the right.

(a) Express z^5 in polar form. (1 mark)



(b) On the above set of axes, plot the following complex numbers. Label them clearly. (3 marks)

(i) z^2

(ii) z^{10}

(c) Determine n such that $z^5 = z^n$ where n $(\neq 5)$ is a positive integer. (3 marks)

Question 17 **(9 marks)**

A line ℓ passes through the z-axis at $z = 2$ and is parallel to the y-axis.

(a) State the vector equation of ℓ. (2 marks)

(b) Find the vector equation of the plane π that contains ℓ and passes through (1, 0, 0). (3 marks)

(c) The plane π in part (b) touches a sphere whose centre is at (0, 0, 0). Find the Cartesian equation of the sphere. (4 marks)

Question 18 **(10 marks)**

The position of a particle P is defined by the vector function

$$\mathbf{r}(t) = (-1 + 3\sec bt)\,\mathbf{i} + (2 + 4\tan bt)\,\mathbf{j}$$

where $b > 0$ and $0 \le t \le \pi$.

(a) Determine the Cartesian equation corresponding to the vector equation. (3 marks)

(b) Find the velocity vector of the particle P at time t. (1 mark)

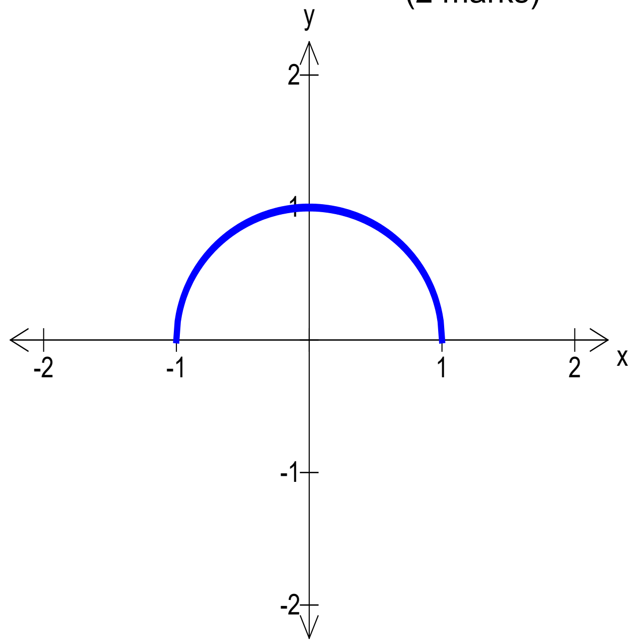
(c) Particle P is moving vertically when $t = \dfrac{\pi}{2}$. Determine the minimum value of b. (3 marks)

A second particle Q starts at the same instant as P and moves with position vector given by $\mathbf{r}(t) = (-1 + t)\,\mathbf{i} + (5 - t)\,\mathbf{j}$.

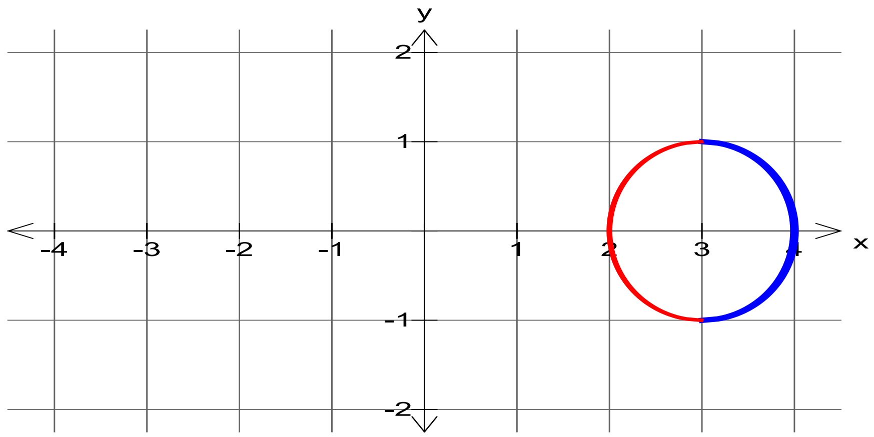
(d) Assuming the minimum value of b in part (c), determine whether P and Q collide or their paths cross. State the time when the incident happens. (3 marks)

Question 19 **(9 marks)**

(a) By considering the **geometric** interpretation of $\int_{-1}^{1}\sqrt{1-x^2}\,dx$ **explain** why $\int_{-1}^{1}\sqrt{1-x^2}\,dx = \frac{1}{2}\pi$. (2 marks)



(b) A circular disc with equation $(x - 3)^2 + y^2 = 1$ is rotated about the **y-axis**. The solid obtained is a torus (a donut shaped solid).
Find, by integration, the exact volume of the solid. (7 marks)



Question 20 **(6 marks)**

Consider a **real** coefficient polynomial $P(z) = z^3 - 4z^2 + cz + d$ where c and d are unknowns.

(a) Given that $z = u$ ($\neq 0$) is a root of the equation $z^3 - 4z^2 + cz + d = 0$, determine one solution of the equation $dz^3 + cz^2 - 4z + 1 = 0$ in terms of u. (3 marks)

(b) Given that $z = 1 + i$ is a root of the equation $z^3 - 4z^2 + cz + d = 0$, determine the values of c and d. (3 marks)

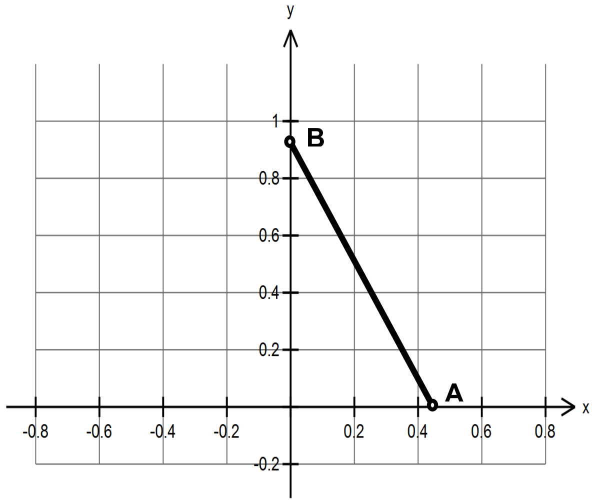
Question 21 **(7 marks)**

The end points of a movable rod AB of length 1 metre have coordinates (x, 0) and (0, y). The position of the end point A on the x-axis is

$$x(t) = \frac{1}{2}\sin\frac{\pi t}{6}$$

where t is the time in seconds.

(a) Determine the period of point B. (2 marks)



(b) Determine the maximum speed of B and the first time it happens. (5 marks)

Additional working space

Question number(s): ________________

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